Name: ______________________________ GTID: ______________________________

- Fill out your name and Georgia Tech ID number.
- This quiz contains 5 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the TA when handing in.
- Please write detailed solutions including all steps and computations.
- The duration of the quiz is 30 minutes.

Good luck!

1. (60 points) Find the general solution of $\mathbf{X}' = \begin{pmatrix} 1 & 6 \\ 2 & -3 \end{pmatrix} \mathbf{X}$.

Solution: Eigenvalues of our matrix satisfy:

$$| \mathbf{A} - \lambda \mathbf{I} | = \begin{vmatrix} 1-\lambda & 6 \\ 2 & -3-\lambda \end{vmatrix} = \mathbf{0}$$

which means that

$$-3 + 2\lambda + \lambda^2 - 12 = 0,$$

meaning that

$$(\lambda + 1)^2 = 16$$

$\iff \lambda_1 = 3, \lambda_2 = -5 \rightarrow$ 24 points

Eigenvectors for $\lambda_1 = 3 : (\mathbf{A} - 3\mathbf{I})\mathbf{X} = \mathbf{0} \iff$

$$\begin{pmatrix} -2 & 6 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_2 = \frac{1}{3} v_1$$

$$\text{When } v_1 = 3, \text{ the eigenvector } = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \text{12 points}$$

Eigenvectors for $\lambda_2 = -5 : (\mathbf{A} + 5\mathbf{I})\mathbf{X} = \mathbf{0} \iff$

$$\begin{pmatrix} 6 & 6 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff w_2 = -w_1$$

$$\text{When } w_1 = 1, \text{ the eigenvector } = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow \text{12 points}$$

So, the general solution is :

$$\mathbf{X} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} \rightarrow \text{12 points}$$

2. (optional – 10 points) The functions $y_1(t) = e^t - 2e^{2t}$ and $y_2(t) = -3e^t + 6e^{2t}$ are solutions of a linear second order ODE. Are these solutions linearly independent?

Solution: No, since for $t = 0$ the Wronskian of these solutions is the determinant of the matrix $\begin{pmatrix} -1 & 3 \\ -3 & 9 \end{pmatrix}$ and this determinant is equal to $-9 + 9 = 0$.

3. (40 points) Consider the differential equation

$$ty''(t) + 6y(t) = e^t.$$

Rewrite this equation as an equivalent linear first order system of differential equations.

Solution: We introduce a new vector-valued variable $z = (z_1, z_2) := (y, y')$ and calculate $z_1' = y' = z_2$ and

$$z_2' = y'' = \frac{6}{t}y + e^t = -\frac{6}{t}z_1 + e^t.$$

Using these equations we write the system as

$$z' = \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_2 \\ -\frac{6}{t}z_1 + e^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{6}{t} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{6}{t} & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

4. (Bonus – 10 points) Let A be a real-valued matrix. Consider the first order system of differential equations

$$x' = A(t)x.$$

Let y be a (potentially non-real) solution[1]. Why is $\mathrm{Re}\{y\}$ also a solution to the system?

Solution: We know that y solves the system, i.e. $y'(t) = Ay(t)$. If we now analyze the real part of the equation, we get

$$\mathrm{Re}\{y'(t)\} = \mathrm{Re}\{Ay(t)\} = A\,\mathrm{Re}\{y(t)\}$$

because A is a real-valued matrix. We observe that $\mathrm{Re}\{y'(t)\} = \mathrm{Re}\{y\}'(t)$ and $\mathrm{Re}\{y(t)\} = \mathrm{Re}\{y\}(t)$. In total, we get

$$\mathrm{Re}\{y\}'(t) = A\,\mathrm{Re}\{y\}(t)$$

which by definition means that $\mathrm{Re}\{y\}$ is a solution to the differential equation.

[1] y being a non-real solution means that for some t_0 the value $y(t_0)$ is a complex but not real number.